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THE BITANGENTIAL OF THE QUINTIC.

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The problem of finding a curve which shall pass through the points of contact of the bitangents of a given curve has engaged the attention of the most eminent mathematicians. Salmon, in his Higher Plane Curves, gives two methods of finding the equation of the bitangential curve, but the application of these methods is attended with difficulties which have not been overcome except for the single case of the curve of the fourth order. The equation of the bitangential of the quartic is given by Salmon in a very elegant form, but in attempting to apply the same methods to the quintic he was not successful. He says (Higher Plane Curves, 3d edition, p. 351), "In order to form the bitangential curve of a quintic the quantity to be calculated is

$$(27Q_2Q_5 - 5Q_3Q_4)^2 = 5(4Q_3^2 - 9Q_2Q_4)(5Q_4^2 - 12Q_3Q_5)$$

a quantity containing $\alpha\beta\gamma$ in the sixth order, and which it is necessary, by the help of the equation of the curve, to show to be divisible by R^6. Now, in virtue of a formula already obtained, we have

$$4Q_3^2 - 9Q_2Q_4 = R^2(4\theta - 9H\Phi).$$

It is also easy to show that $27Q_2Q_5 - 5Q_3Q_4$ and $5Q_4^2 - 12Q_3Q_5$ are each divisible by R; but I have not been able to carry the reduction further."

And again (p. 357, foot note): "I attempted in like manner to obtain the bitangential curve of a quintic by writing down for the curve, whose equation is given Art. 394, a covariant of the right order and such that the absolute term vanished if the axis of x touches the given curve a second time. For instance, if $\psi = 4\theta - 9H\Phi$, then $A\left(\frac{d\psi}{dx}\right)^2 + \dots$ and $\psi\left(A\frac{d^2\psi}{dx^2} + \dots\right)$ are covariants of the right order. Although I have not been successful, it may be useful for purposes of reference to give the values I obtained for the covariants in this case.

It will be seen that, without loss of generality, we may suppose c_1 and c_2 to vanish. We have, then,

$$\begin{aligned} H = \; & b^2c + 3b^2(d_0x + d_1y) \\ & + 3(b^2e_0 - 4bcd_1)x^2 + 3(2b^2e_1 - 5bcd_2)xy \\ & + 3(b^2e_2 - bcd_3)y^2 + (b^2f_0 - 16bce_1 + 18c^2d_2)x^3 \\ & + (3b^2f_1 - 39bce_2 - 9bd_0d_2 + 9bd_1^2 + 18c^2d_3)x^2y \\ & + (-6bcf_1 - 12bd_0e_1 + 12be_0d_1 + 18c^2e_2 + 24cd_0d_2 - 18cd_1^2)x^4 \\ & + \ldots, \end{aligned}$$

$$\Theta = 9b^2 \left\{ \begin{array}{l} b^4d_0^2 + 6b^3c^2d_1 \\ + \; (4b^4d_0e_0 + 12b^3c^2e_1 - 6b^3cd_0d_1 - 57b^2c^3d_2)x \\ + \left\{ \begin{array}{l} 2b^4d_0f_0 + 4b^4e_0^2 + 6b^3c^2f_1 + 6b^3cd_0e_1 - 48b^3cd_1e_0 \\ - 105b^2c^3e_2 - 293b^2c^2d_0d_2 + 269b^2c^2d_1^2 + 36bc^4d_3 \end{array} \right\} x^2 \\ + \ldots \end{array} \right\},$$

$$\Phi = 6b \left\{ \begin{array}{l} b^3e_0 + 4b^2cd_1 + (b^3f_0 - 8b^2ce_1 - 38bc^2d_2)x \\ + [b^3f_1 - 2b^2ce_2 + 27b^2(d_1^2 - d_0d_2) - 41bc^2d_3]y \\ + \left\{ \begin{array}{l} - 12b^2cf_1 - 12b^2d_0e_1 + 12b^2e_0d_1 + 6bc^2e_2 \\ - 162bcd_0d_2 + 168bcd_1^2 - 6c^3d_3 \end{array} \right\} x^2 \\ + \ldots \end{array} \right\}.$$

Of the quantities A, B, etc., the only ones which contain terms independent of x and y are $A = b^2$, $F = bc$; so that if any quantity ψ of the form $\Theta + kH\Phi$ written at full length be

$$A + B_0x + B_1y + C_0x^2 + \ldots,$$

then the degree of ψ being 22, the absolute term in the covariant $A\left(\frac{d\psi}{dx}\right)^2 + \ldots$ is $b^2B_0^2 + 44bcAB_1$, and in $A\frac{d^2\psi}{dx^2} + \ldots$ is $2b^2C_0 + 42bcB_1$."

I quote thus in full for the reason that I have recently discovered the functions sought and have verified the result by the method indicated by Salmon.

I gave a form for the bitangential of a quintic in the Analyst, Vol. VIII, p. 171. The equation was obtained by analogy from the equation of the bitangential of the quartic; but I stated that I had not verified the result, and the results of the present paper show that it was not the true form.

Let us take the origin on the curve, and the axis of x as the tangent. The equation of the curve may then be written in the form

$$\begin{aligned} u = {} & 5by + 10(c_0x^2 + 2c_1xy + c_2y^2) \\ & + 10(d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) \\ & + \ 5(e_0x^4 + 4e_1x^3y + 6e_2x^2y^2 + 4e_3xy^3 + e_4y^4) \\ & + (f_0x^5 + 5f_1x^4y + 10f_2x^3y^2 + 10f_3x^2y^3 + 5f_4xy^4 + f_5y^5) \\ = {} & 0. \end{aligned}$$

I prefer to work with this form, which is more general, than with that used by Salmon, although my results were first obtained by the use of his equations given in the foot note before mentioned.

The points in which the axis of x meets the curve are found by making $y = 0$ in this equation, and are therefore given by the equation

$$x^2(f_0x^3 + 5e_0x^2 + 10d_0x + 10c_0) = 0.$$

Two of these points coincide with the origin, as they evidently should, the axis being a tangent. Two other points in which the axis meets the curve will coincide, or the axis will be a double tangent if the discriminant of

$$f_0x^3 + 5e_0x^2 + 10d_0x + 10c_0 = 0$$

vanishes; that is, if we have

$$27c_0^2f_0^2 + 40d_0^3f_0 + 50c_0e_0^3 - 25d_0^2e_0^2 - 90c_0d_0e_0f_0 = 0.$$

We now write for the second differential coefficients of u

$$a = \frac{d^2u}{dx^2} = 20\left\{\begin{array}{l} c_0 + 3(d_0x + d_1y) + 3(e_0x^2 + 2e_1xy + e_2y^2) \\ \quad + (f_0x^3 + 3f_1x^2y + 3f_2xy^2 + f_3y^3) + \dots \end{array}\right\},$$

$$b = \frac{d^2u}{dy^2} = 20\left\{\begin{array}{l} c_2 + 3(d_2x + d_3y) + 3(e_2x^2 + 2e_3xy + e_4y^2) \\ \quad + (f_2x^3 + 3f_3x^2y + 3f_4xy^2 + f_5y^3) + \dots \end{array}\right\};$$

$$c = \frac{d^2u}{dz^2} = 20\left\{\begin{array}{l} 3by + 3(c_0x^2 + 2c_1xy + c_2y^2) \\ \quad + (d_0x^3 + 3d_1x^2y + 3d_2xy^2 + d_3y^3) + \dots \end{array}\right\},$$

$$f = \frac{d^2u}{dydz} = 20\left\{\begin{array}{l} b + 3(c_1x + c_2y) + 3(d_1x^2 + 2d_2xy + d_3y^2) \\ \quad + (e_1x^3 + 3e_2x^2y + 3e_3xy^2 + e_4y^3) + \dots \end{array}\right\},$$

$$g = \frac{d^2u}{dzdx} = 20\left\{\begin{array}{l} 3(c_0x + c_1y) + 3(d_0x^2 + 2d_1xy + d_2y^2) \\ \quad + (e_0x^3 + 3e_1x^2y + 3e_2xy^2 + e_3y^3) + \dots \end{array}\right\},$$

$$h = \frac{d^2u}{dxdy} = 20 \left\{ \begin{array}{l} c_1 + 3(d_1x + d_2y) + 3(e_1x^2 + 2e_2xy + e_3y^2) \\ + (f_1x^3 + 3f_2x^2y + 3f_3xy^2 + f_4y^3) + \dots \end{array} \right\}.$$

If we now write

$$\Sigma' = \left| \begin{array}{cccc} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{array} \right|,$$

we have

$$\Sigma' = A\alpha^2 + B\beta^2 + C\gamma^2 + 2F\beta\gamma + 2G\gamma\alpha + 2H\alpha\beta,$$

where

$$\begin{aligned} A &= bc - f^2 \\ &= -b^2 - 6bc_1x - 3bc_2y + 3(c_0c_2 - 2bd_1 - 3c_1^2)x^2 \\ &\quad - 3(bd_2 + 4c_1c_2)xy + 3(bd_3 - 2c_2^2)y^2 \\ &\quad + (c_2d_0 - 2be_1 + 9c_0d_2 - 9c_1d_1)x^3 \\ &\quad + 3(be_2 + 3c_0d_3 - 5c_2d_1 - 6c_1d_2)x^2y \\ &\quad + 12(be_3 - 2c_2d_2)xy^2 + (7be_4 - 8c_2d_3)y^3 \\ &\quad + \dots, \end{aligned}$$

$$B = ac - g^2 = 3bc_0y + \dots,$$

$$\begin{aligned} C = ab - h^2 &= (c_0c_2 - c_1^2) + 3(c_0d_2 + c_2d_0 - 2c_1d_1)x \\ &\quad + 3(c_0d_3 + c_2d_1 - 2c_1d_2)y + \dots, \end{aligned}$$

$$F = gh - af = -bc_0 - 3bd_0x + 3(c_1^2 - c_0c_2 - bd_1)y + \dots,$$

$$\begin{aligned} G = hf - bg &= bc_1 + 3(c_1^2 + bd_1 - c_0c_2)x + 3bd_2y \\ &\quad + 3(be_1 - c_2d_0 - 3c_0d_2 + 4c_1d_1)x^2 \\ &\quad + 3(c_2d_1 + 2be_2 + 2c_1d_2 - 3c_0d_3)xy \\ &\quad + 3(be_3 - 2c_1d_3 + 2c_2d_2)y^2 \\ &\quad + (bf_1 - c_2e_0 + 9d_1^2 - 9c_0e_2 - 9d_0d_2 + 10c_1e_1)x^3 \\ &\quad + 3(bf_2 + 2c_2e_1 + 3d_1d_2 - 3d_0d_3 + 4c_1e_2 - 6c_0e_3)x^2y \\ &\quad + 3(bf_3 - 2c_1e_3 + 3d_2^2 - 3d_1d_3 - 3c_0e_4 + 5c_2e_2)xy^2 \\ &\quad + (bf_4 - 8c_1e_4 + 8c_2e_3)y^3 \\ &\quad + \dots, \end{aligned}$$

$$\begin{aligned} H = fg - ch = {} & 3bc_0x + 3(bd_0 + 2c_0c_1)x^2 \\ & + 3(c_1^2 - bd_1 + 3c_0c_2)xy + 6(c_1c_2 - bd_2)y^2 \\ & + (be_0 + 8c_1d_0)x^3 \\ & + 3(2c_1d_1 - 2be_1 + 3c_0d_2 + 3c_2d_0)x^2y \\ & + 3(2c_1d_2 + 3c_0d_3 + 3c_2d_1 - 5be_2)xy^2 \\ & + 8(c_1d_3 - be_3)y^3 \\ & + \ldots, \end{aligned}$$

the Hessian is

$$\begin{aligned} H = {} & \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} \\ = {} & aA + hH + gG. \end{aligned}$$

Introducing the values of a, h, g, A, H, G we have for the equation of the Hessian

$$\begin{aligned} H = {} & b^2c_0 + 3b^2d_0x + 3(bc_0c_2 - bc_1^2 + b^2d_1)y \\ & + 3(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x^2 \\ & + 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)xy \\ & + 3(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1)y^2 \\ & + (b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_0 - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1)x^3 \\ & + 3\left\{ \begin{array}{l} b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 \\ + 6c_0^2d_3 + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{array} \right\} x^2y \\ & + 3\left\{ \begin{array}{l} b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 \\ + 6c_0c_1d_3 + 6c_2^2d_0 + 6bc_2e_1 - 7bc_0e_3 - 14c_1^2d_2 \end{array} \right\} xy^2 \\ & + \left\{ \begin{array}{l} b^2f_3 + 7bc_1e_3 - 7bc_0e_4 - 8c_1^2d_3 + 8c_0c_2d_3 + 9bd_2^2 \\ + 9bc_2e_2 - 9bd_1d_3 + 18c_2^2d_1 - 18c_1c_2d_2 \end{array} \right\} y^3 \\ & + \ldots. \end{aligned}$$

If we now write for α, β, γ the differential coefficients of H with respect to x, y, z in the expression for Σ, we have the covariant θ. Thus

$$\theta = \begin{vmatrix} a, & h, & g, & \dfrac{dH}{dx} \\ h, & b, & f, & \dfrac{dH}{dy} \\ g, & f, & c, & \dfrac{dH}{dz} \\ \dfrac{dH}{dx}, & \dfrac{dH}{dy}, & \dfrac{dH}{dz}, & 0 \end{vmatrix}$$

$$= A\left(\frac{dH}{dx}\right)^2 + B\left(\frac{dH}{dy}\right)^2 + C\left(\frac{dH}{dz}\right)^2$$

$$+ 2F\left(\frac{dH}{dy}\right)\left(\frac{dH}{dz}\right) + 2G\left(\frac{dH}{dz}\right)\left(\frac{dH}{dx}\right) + 2H\left(\frac{dH}{dx}\right)\left(\frac{dH}{dy}\right).$$

We easily find from the value given for H

$$\begin{aligned}\frac{dH}{dx} = {} & 3b^2d_0 + 6(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x \\ & + 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)y \\ & + 3(b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_1 - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1)x^2 \\ & + 6\left\{\begin{matrix} b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 + 6c_0^2d_3 \\ + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{matrix}\right\}xy \\ & + \ldots,\end{aligned}$$

$$\begin{aligned}\frac{dH}{dy} = {} & 3(bc_0c_2 - bc_1^2 + b^2d_1) + 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)x \\ & + 6(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1)y \\ & + 6\left\{\begin{matrix} b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 + 6c_0c_1d_3 \\ + 6c_2^2d_0 + 6bc_2e_1 - 7bc_0e_3 - 14c_1^2d_2 \end{matrix}\right\}xy \\ & + \ldots,\end{aligned}$$

$$\begin{aligned}\frac{dH}{dz} = {} & 9b^2c_0 + 24b^2d_0x + 24(bc_0c_2 - bc_1^2 + b^2d_1)y \\ & + 21(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x^2 \\ & + 21(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)xy \\ & + \ldots.\end{aligned}$$

Whence the value of θ is

$$\theta = (9b^6d_0^2 + 27b^4c_0^2c_1^2 - 27b^4c_0^3c_2 + 54b^5c_0^2d_1 - 54b^5c_0c_1d_0)$$
$$+ \left\{\begin{matrix} - 27b^4c_0^2c_2d_0 + 36b^6d_0e_0 + 54b^5c_1d_0^2 - 54b^5c_0d_0d_1 + 108b^5c_0^2e_1 \\ - 108b^5c_0c_1e_0 - 486b^4c_0c_1^2d_0 - 513b^4c_0^3d_2 + 1026b^4c_0^2c_1d_1 \end{matrix}\right\} x$$
$$+ \ldots .$$

Also, if we write differential symbols for α, β, γ in the expression for Σ, and operate on H, we get the covariant φ. Thus

$$\Phi = \begin{vmatrix} a, & h, & g, & \dfrac{d}{dx} \\ h, & b, & f, & \dfrac{d}{dy} \\ g, & f, & c, & \dfrac{d}{dz} \\ \dfrac{d}{dx}, & \dfrac{d}{dy}, & \dfrac{d}{dz}, & 0 \end{vmatrix} H$$

$$= A\frac{d^2H}{dx^2} + B\frac{d^2H}{dy^2} + C\frac{d^2H}{dz^2} + 2F\frac{d^2H}{dydz} + 2G\frac{d^2H}{dzdx} + 2H\frac{d^2H}{dxdy}.$$

We easily find

$$\frac{d^2H}{dx^2} = 6(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)$$
$$+ 6\left\{\begin{matrix} b^2f_0 + 8c_0c_2d_0 + 10c_1^2d_0 + 16bc_1e_1 \\ - 16bc_0e_1 + 18c_0^2d_2 - 36c_0c_1d_1 \end{matrix}\right\} x$$
$$+ \ldots,$$

$$\frac{d^2H}{dxdy} = 3(2bc_1d_1 + 2b^2e_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)$$
$$+ 6\left\{\begin{matrix} b^2f_1 + 3bd_1^2 + 3bc_2e_0 - 3bd_0d_2 - 4c_0c_2d_1 + 6c_0^2d_3 \\ + 10bc_1e_1 + 12c_1c_2d_0 - 13bc_0e_2 - 14c_1^2d_1 \end{matrix}\right\} x$$
$$+ \ldots,$$

$$\frac{d^2H}{dxdz} = 24b^2d_0 + 42(b^2e_0 + 2c_0^2c_2 - 2c_0c_1^2 + 4bc_1d_0 - 4bc_0d_1)x$$
$$+ \ldots,$$

$$\begin{aligned}\frac{d^2H}{dy^2} &= 6(b^2e_2 - bc_0d_3 + 2c_0c_2^2 - 2c_1^2c_2 - 2bc_1d_2 + 3bc_2d_1) \\ &\quad + 6\left\{\begin{matrix} b^2f_2 - bc_1e_2 + 2c_0c_2d_2 + 3bd_1d_2 - 3bd_0d_3 + 6c_0c_1d_3 \\ + 6c_2^2d_0 + 6bc_2c_1 - 7bc_0e_3 - 14c_1^2d_2 \end{matrix}\right\}x \\ &\quad + \dots, \\ \frac{d^2H}{dydz} &= 24(bc_0c_2 - bc_1^2 + b^2d_1) \\ &\quad + 21(2bc_1d_1 + 2b^2c_1 + 3bc_2d_0 + 4c_0c_1c_2 - 4c_1^3 - 5bc_0d_2)x \\ &\quad + \dots, \\ \frac{d^2H}{dz^2} &= 72b^2c_0 + 168b^2d_0x + \dots\end{aligned}$$

Whence
$$\begin{aligned}\Phi &= 6(b^4c_0 + 2b^2c_0c_1^2 - 2b^2c_0^2c_2 + 4b^3c_0d_1 - 4b^3c_1d_0) \\ &\quad + \left\{\begin{matrix} 6b^4f_0 + 24b^2c_0c_2d_0 + 48b^3c_1e_0 - 48b^3c_0c_1 \\ - 228b^2c_0^2d_2 - 252b^2c_1^2d_0 + 456b^2c_0c_1d_1 \end{matrix}\right\}x \\ &\quad + \dots\end{aligned}$$

We will now show that the equation of the bitangential is

$$[297HJ(u, H, \Phi) - 8208HJ'(u, H, \Phi) - 20J(u, H, \Theta)]^2 = 75625(4\Theta - 9H\Phi)^3,$$

where by $J(u, H, \varphi)$ we mean the Jacobian of these three functions; and $J'(u, H, \varphi)$ means that, in forming the Jacobian, φ is to be differentiated on the supposition that the second differential coefficients of H, which enter into the expression for φ, are constant.

We verify this result by showing that the absolute term vanishes.

We find without difficulty the absolute term in

$$4\Theta - 9H\Phi = 36b^6d_0^2 - 54b^6c_0e_0 = 18b^6(2d_0^2 - 3c_0e_0).$$

To calculate the absolute term in J it is useful to observe that, if F is any function of x, y, z, the absolute term in $J(u, H, F)$ is

$$\begin{vmatrix} \frac{du}{dx}, & \frac{du}{dy}, & \frac{du}{dz} \\ \frac{dH}{dx}, & \frac{dH}{dy}, & \frac{dH}{dz} \\ \frac{dF}{dx}, & \frac{dF}{dy}, & \frac{dF}{dz} \end{vmatrix} = \begin{vmatrix} 0, & 5b, & 0 \\ 3b^2d_0, & \frac{dH}{dy}, & 9b^2c_0 \\ \frac{dF}{dx}, & \frac{dF}{dy}, & \frac{dF}{dz} \end{vmatrix}$$

$$= 5b\left\{9b^2c_0\frac{dF}{dx} - 3b^2d_0\frac{dF}{dz}\right\};$$

and similarly for J'.

Whence we have after reduction

$$J(u,\ H,\ \varphi) = 5b \left\{ \begin{array}{l} 54b^6c_0f_0 - 234b^6d_0e_0 + 432b^5c_0c_1e_0 \\ -432b^5c_0^2e_1 + 684b^4c_0^2c_2d_0 + 864b^5c_0c_1e_1 \\ +936b^5c_1d_0^2 - 936b^5c_0d_0d_1 - 2052b^4c_0^3d_2 \\ -2736b^4c_0c_1^2d_0 + 4104b^4c_0^2c_1d_1 \end{array} \right\} + \dots,$$

$$J(u,\ H,\ \theta) = 5b \left\{ \begin{array}{l} 324b^8c_0d_0e_0 - 594b^8d_0^3 + 972b_0^7c^3e_1 \\ -972b^7c_0^2c_1e_0 + 1539b^6c_0^3c_2d_0 + 4050b^7c_0c_1d_0^2 \\ -4050b^7c_0^2d_0d_1 - 4617b^6c_0^4d_2 - 6156b^6c_0^2c_1^2d_0 \\ +9234b^6c_0^3c_1d_1 \end{array} \right\} + \dots,$$

$$J'(u,\ H,\ \varphi) = 5b \left\{ \begin{array}{l} -6b^6d_0e_0 + 18b^5c_0c_1e_0 - 18b^5c_0^2e_1 \\ +21b^4c_0^2c_2d_0 + 24b^5c_1d_0^2 - 24b^5c_0d_0d_1 \\ -63b^4c_0^3d_2 - 84b^4c_0c_1^2d_0 + 126b^4c_0^2c_1d_1 \end{array} \right\} + \dots.$$

Whence we find the absolute term in

$$297HJ(u,\ H,\ \varphi) - 8208HJ'(u,\ H,\ \varphi) - 20J(u,\ H,\ \theta)$$

to be $$2970b^9(27c_0^2f_0 - 45c_0d_0e_0 + 20d_0^3).$$

Therefore the absolute term in

$$[297HJ(u,\ H,\ \Phi) - 8208HJ'(u,\ H,\ \Phi) - 20J(u,\ H,\ \theta)]^2 - 75625(4\theta - 9H\Phi)^3 = 0.$$

is proportional to

$$b^{18}c_0^2(27c_0^2f_0^2 + 40d_0^3f_0 + 50c_0e_0^3 - 25d_0^2e_0^2 - 90c_0d_0e_0f_0),$$

and therefore vanishes, if $y = 0$ is a double tangent. That is, this curve passes through the points of contact of the double tangents.

It should be observed that the equation we have found for the bitangential contains an irrelevant factor of the eighteenth order in the variables and of the sixth in the coefficients of the given curve. This factor appears to be the square of the Hessian, and dividing out, the resulting equation is of the forty-eighth order in the variables and of the eighteenth in the coefficients of the original equation.